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Obviously cyclical permutation of the symbols leads to the required proportions

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4.$$

Remark: From the principles of statics we know that the four original scalar equations must be compatible or simultaneous and, from the theory of determinants, that the necessary and sufficient condition for this compatibility is the vanishing of the determinant Δ of the coefficients of W, X, Y, and Z. Hence, the six cosines l, m, n, p, q, r pertaining to a closed space quadrilateral are not all independent but are mutually connected by the relation $\Delta = 0$, that is,

$$l^2 + m^2 + n^2 + p^2 + q^2 + r^2 - l^2r^2 - m^2q^2 - n^2p^2 - 2lmp$$
$$- 2lnq - 2mnr - 2pqr + 2lmqr + 2lnpr + 2mnpq = 1.$$

310. Proposed by EMMA M. GIBSON, Drury College.

"A particle moveable on a smooth spherical surface of radius a is projected along the horizontal great circle with a velocity v which is great compared with $\sqrt{(2ga)}$. Prove that its path lies between this great circle and a parallel circle whose plane is at a depth $2ga^2/v^2$ below the centre, approximately."

From Lamb's *Dynamics*, p. 334, Ex. 3.

SOLUTION BY H. S. UHLER, Yale University.

Since the particle has only two degrees of freedom let us choose as coördinates the angles θ and ϕ which the radius terminating in the particle (of mass m) makes with any fixed vertical plane which contains the center of the sphere and with the horizontal plane passing through this center, respectively. Since none of the forces acting on the particle has a moment about the vertical diameter of the sphere it follows that the moment of momentum about this axis must remain invariable. At the instant of projection the moment of momentum equals $a \cdot mv$. At some later time the moment of momentum will be $r \cdot mr\dot{\theta}$ or $ma^2(\cos^2 \phi)\dot{\theta}$, because $r = a \cos \phi$. Consequently

$$a(\cos^2 \phi)\dot{\theta} = v. \tag{1}$$

Another relation between θ and ϕ may be derived from the principle of the conservation of energy. At the instant of projection the total energy equals $\frac{1}{2}mv^2 + V$, where V symbolizes the potential energy at the level of the center of the sphere. In general, at any later time the components of the kinetic energy in the horizontal and vertical planes will be respectively $\frac{1}{2}ma^2(\cos^2 \phi)\dot{\theta}^2$ and $\frac{1}{2}ma^2\dot{\phi}^2$, while the potential energy will be $V - mga \sin \phi$. Therefore,

$$a^2(\cos^2 \phi)\dot{\theta}^2 + a^2\dot{\phi}^2 - 2ag \sin \phi = v^2. \tag{2}$$

Substituting the value of $\dot{\theta}$ from (1) in (2) we find

$$a^2\dot{\phi}^2 = 2ag \sin \phi - v^2 \tan^2 \phi$$

or

$$a\dot{\phi} = (2ag \sin \phi - v^2 \tan^2 \phi)^{\frac{1}{2}}.$$

The last equation shows that ϕ cannot be negative with $\dot{\phi}$ real, in other words, the particle will not rise above the plane of projection. Writing this equation in the form

$$2a\sqrt{2ag}(\cos \phi)\dot{\phi} = [(\sin \phi)(\sqrt{v^4 + 16a^2g^2} + v^2 + 4ag \sin \phi)(\sqrt{v^4 + 16a^2g^2} - v^2 - 4ag \sin \phi)]^{\frac{1}{2}},$$

it becomes self-evident that the greatest value which ϕ can attain without making $\dot{\phi}$ complex is given by the formula

$$4ag \sin \phi_0 = \sqrt{v^4 + 16a^2g^2} - v^2.$$

The corresponding vertical distance below the horizontal diametral plane is $a \sin \phi_0 = d$. Then, rigorously,

$$d = \frac{1}{4g}(\sqrt{v^4 + 16a^2g^2} - v^2).$$

When $4ag/v^2$ is less than unity we may expand the radical by the binomial theorem to obtain an approximate rational expression for d. Then

$$d = \frac{v^2}{4g}\left[\left(1 + \frac{8a^2g^2}{v^4} - \frac{32a^4g^4}{v^8} + \cdots\right) - 1\right]$$

or, as a first approximation,

$$d = 2ga^2/v^2.$$

Remark: If $v = 0$, (1) gives $\dot{\theta} = 0$ or $\theta = constant$, showing that the motion is now uniplanar. Under these conditions (2) reduces to $a\dot{\phi}^2 = 2g \sin \phi$, since $\dot{\phi} \neq 0$. Then $a\ddot{\phi} = g \cos \phi$ or putting $\phi = (\pi/2) + \xi$, $a\ddot{\xi} = - g \sin \xi$ which is the familiar form of the equation of motion of a simple pendulum.

311. Proposed by B. J. BROWN, Student in Drury College.

A particle oscillates in a straight line about a center of force varying as the distance, and is subject to a retardation $k \times (\text{vel.})^2$. If a, b be two successive elongations, on opposite sides, prove that

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

What form does the result take if a is infinite?

From Lamb's *Dynamics*, p. 299, ex. 14.

I. SOLUTION BY H. S. UHLER, Yale University.

Let the displacement of the particle at any time t from the center of attraction be denoted by x. To fix the ideas, consider the forces acting on the particle of mass m at an instant when its displacement and velocity are both positive. Then

$$m\frac{d^2x}{dt^2} = - cmx - km\left(\frac{dx}{dt}\right)^2,$$

where c is a positive constant, or

$$\frac{d^2x}{dt^2} + k\left(\frac{dx}{dt}\right)^2 + cx = 0. \tag{1}$$

Equation (1) is a standard form of which the first integral can be obtained at once, for the solution of

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 f(y) + F(y) = 0$$

is

$$\frac{dy}{dx}e^{\int f(y)dy} = \left\{A - 2\int dyF(y)e^{2\int f(y)dy}\right\}^{\frac{1}{2}}.$$

See Forsyth's *Differential Equations*, third ed., Art. 67.

Hence

$$\frac{dx}{dt}e^{kx} = \left\{A + \frac{ce^{2kx}}{2k^2}(1 - 2kx)\right\}^{\frac{1}{2}}, \tag{2}$$

The constant of integration A in equation (2) may be found from the datum that $dx/dt = 0$ when $x = - a$, whence

$$\frac{dx}{dt}e^{kx} = \frac{1}{k}\left\{\frac{c}{2}\left[(1 - 2kx)e^{2kx} - (1 + 2ka)e^{-2ka}\right]\right\}^{\frac{1}{2}}. \tag{3}$$

Again, by hypothesis, $dx/dt = 0$ when $x = b$ so that equation (3) gives the required relation

$$(1 + 2ka)e^{-2ka} = (1 - 2kb)e^{2kb}.$$

When a becomes infinite we have the so-called indeterminate form

$$\frac{1 + 2ka}{e^{2ka}} = 0 + \frac{z}{e^z} = \frac{\infty}{\infty}.$$

In general,

$$\frac{z}{e^z} = \left[z^{-1} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots + \frac{z^n}{(n+1)!} + \cdots\right]^{-1};$$

consequently as z increases indefinitely ze^{-z} approaches zero as a limit. The same result follows from the well-known fact that "any infinite number is an infinity of higher order than any power of its logarithm."